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EXCELLENT RINGS, HENSELIAN RINGS, AND THE APPROXIMATION PROPERTY

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1. Excellent rings. In the 1950s M. Nagata constructed a variety of local Noetherian rings which behave strangely under completion. For example, he constructed a local Noetherian normal domain $(A, \mathbf{m})$ whose completion $\hat{A}$ is not reduced [**15**, p. 209]. The observation of such deviant behavior within the supposedly "nice" class of Noetherian rings has led to the widespread use of certain additional conditions on Noetherian rings, conditions which insure good behavior for completions. Rings which satisfy these conditions are called "excellent."

Before defining excellence, we consider the question:

Question. What properties should a Noetherian ring have in order to be considered "excellent"?

It has been generally agreed that "excellent" Noetherian rings should behave similarly to the rings found in algebraic geometry, specifically, those rings of the form

$$A = K[X_1, \dots , X_n]/I$$

where A has finite type over a field K. Our question becomes:

Question. What are the fundamental properties of geometric rings?

To consider this question formally, let K be a field, and let $A = K[X_1, \dots , X_n]/I$ be a reduced ring. Then A is the coordinate ring of an algebraic variety $Y = Z(I)$, or the set of zeros of the ideal I in n-dimensional space over K. From a well-known theorem in algebraic

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geometry, the singular locus of Y is again an algebraic variety [**10**, Theorem 5.3], that is,

$$\begin{aligned}\operatorname{Sing}(A) :&= \{P \in \operatorname{Spec}(A) \mid A_P \text{ not regular}\}\\ &= V(J) \text{ for some ideal } J \text{ of } A\end{aligned}$$

and this set is closed in $\operatorname{Spec}(A)$. We list this as the first geometric property:

Property (a). *The singular locus of A is closed in* $\operatorname{Spec}(A)$.

A related property of geometric rings, used in the proof of Property (a), is that these rings admit the "Jacobian criteria." If the field K is algebraically closed, the following equivalences hold:

$$\begin{aligned}A_P \text{ is regular } &\Longleftrightarrow \operatorname{rank}((\partial f_i/\partial x_j) \bmod (P)) = r\\ &\Longleftrightarrow P \notin \text{ the ideal generated by the}\\ &\qquad r \times r \text{ minors of } (\partial f_i/\partial X_j).\end{aligned}$$

Here r denotes the height of the ideal $I = (f_1, \dots, f_s)$ in the localized polynomial ring $K[X_1, \dots, X_n]_P$. If the field K is not perfect, the Jacobian criterion stated above does not measure regularity but does measure smoothness, which is not equivalent to regularity. For example, let K be a field of characteristic $p > 0$, and let $L = K[X]/(f)$, where $f = X^p - a$ for some element $a \in K - K^p$. Then L is a proper purely inseparable extension of K. Since $\partial f/\partial X = 0$ the Jacobian criterion for smoothness shows that L fails to be smooth over K. However, L is regular and there is a K^p-derivation $D : K[X] \to K[X]$ with $D(f) \neq 0$. This reflects the basic idea of Nagata's Jacobian criterion for regularity [**14**, Theorem 30.10]. If the field K is not perfect of characteristic $p > 0$, the derivations under consideration are not only the partial K-derivations $\partial f_i/\partial X_j$, but also the K^{p^n}-derivations of K (for all $n \in \mathbf{N}$). Of course, over perfect fields the two concepts agree.

Three additional fundamental properties of geometric rings also deserve mention:

Property (b). *Every extension of finite type is a geometric ring.*

Property (c). *A is universally catenary. This means that A is Noetherian and for every finitely generated algebra B over A, whenever P and Q are prime ideals of B with $P \subseteq Q$, every two maximal chains of prime ideals from P to Q have the same length. Catenary conditions are discussed in full detail in [**14**, Section 31].*

Property (d). *Hironaka's theorem on resolution of singularities for schemes over geometric rings holds (at least in characteristic 0).*

Among these properties, the fact that geometric rings admit Jacobian criteria is the most central for the theory of excellent rings. Although some consider resolution of singularities more important, the Jacobian criteria are more relevant to the definition and study of excellent rings. Moreover, it is somewhat misleading to mention Hironaka's theorem in regard to the geometric motivation for excellence. The concept of excellent rings was known at the time of Hironaka's work on the resolution of singularities. Indeed, Hironaka proved resolution of singularities for schemes (of finite type) over excellent rings containing the rationals.

One possible method of enlarging the class of geometric rings would be to include all rings which admit Jacobian criteria. Unfortunately, the class of local Noetherian rings having a sufficiently large module of derivations to permit establishment of Jacobian criteria on the rings is fairly small, and it is too restrictive to consider only this class of rings. This may be illustrated by the following example:

Let $\sigma = \exp(\exp(X) - 1) = e^{(e^X - 1)} \in \mathbf{Q}[[X]]$. The power series σ is chosen such that σ and $\partial\sigma/\partial X$ are algebraically independent over $\mathbf{Q}(X)$ (see [**4**]). The intersection ring $A = \mathbf{Q}(X, \sigma) \cap \mathbf{Q}[[X]]$ is a discrete valuation ring which contains $\mathbf{Q}[X]_{(X)}$ and which has field of quotients $\mathbf{Q}(X, \sigma)$. However, every derivation $d : A \to \mathbf{Q}[[X]]$ satisfies the condition $d(\sigma) = d(X)\partial\sigma/\partial X$. Since A has transcendence degree 2 over $\mathbf{Q}$ and since σ and $\partial\sigma/\partial X$ are algebraically independent over $\mathbf{Q}(X)$, it follows that $d(\sigma) \notin A$ whenever $d(X) \neq 0$. Hence there is only the trivial derivation from A into itself.

But then, what conditions on A would imply a closed singular locus?

The key observation here is that, although there are no Jacobian criteria available for arbitrary Noetherian local rings, there are Jacobian and regularity criteria for complete local rings. Nagata and Grothendieck formulated these criteria, which are similar to the above-mentioned criterion. The main objective in the theory of excellent rings is to make use of the Jacobian criteria on the completion $\hat{A}$ of a local (excellent) ring A in order to describe some of its (A's) properties, although the ring A itself fails to admit Jacobian criteria. This theory requires considerable theoretical background. Grothendieck's theory of formal smoothness was developed to make this connection between the local ring A and its completion $\hat{A}$ work. In the following, we describe briefly its main ideas.

For a local Noetherian ring $(A, \mathbf{m})$, the completion $\hat{A}$ is the homomorphic image of a formal power series ring over K, where K is a field or a complete discrete valuation ring, that is, $\hat{A} \cong K[[X_1, \dots, X_n]]/I$. The singular locus of $\hat{A}$ is closed by the Jacobian criteria on complete local rings [**14**, Corollary to Theorem 30.10]. One way to guarantee closure of the singular locus of A is to require that the singular loci of A and of $\hat{A}$ be generated by the same ideal, that is, to require:

$$\mathrm{Sing}\,(A) = V(J)$$

and

$$\mathrm{Sing}\,(\hat{A}) = V(J\hat{A}), \quad \text{for some ideal } J \subseteq A.$$

This last condition is equivalent to:

$$\forall Q \in \mathrm{Spec}\,(\hat{A}), \quad A_{Q\cap A} \text{ is regular} \iff \hat{A}_Q \text{ is regular.}$$

Since the induced morphism $A_{Q\cap A} \to \hat{A}_Q$ is faithfully flat, the direction "$\Leftarrow$" is always satisfied. If the fiber at $P = Q \cap A$ (the ring $\hat{A}_Q/P\hat{A}_Q$) is regular, then "$\Rightarrow$" holds.

We still require a few more preliminaries for the definition of an "excellent" Noetherian ring:

Definition 1.1. Let $(A, \mathbf{m})$ be a local Noetherian ring and $\hat{A}$ its completion. The *formal fibers* of A are the rings $\hat{A} \otimes_A k(P)$ where $P \in \mathrm{Spec}\,(A)$ and $k(P) = A_P/PA_P$.

In a local Noetherian ring, if the formal fibers $\hat{A} \otimes_A k(P)$ are regular, then A has an open regular locus (the complement of the singular locus). Since an open regular locus is desirable in any algebra of finite type over A, we need a more complex definition:

Definition 1.2. Let K be a field and B a Noetherian K-algebra. B is called *geometrically regular* over K if, for every finite field extension $K \subseteq L$, the ring $B \otimes_K L$ is regular.

Remark 1.3. We can restrict to purely inseparable finite field extensions $K \subseteq L$ in Definition 1.2.

We now state the main definition:

Definition 1.4 (Grothendieck). Let $(A, \mathbf{m})$ be a local Noetherian ring. Then A is called *excellent* if

(a) The formal fibers of A are geometrically regular, that is, for all $P \in \operatorname{Spec}(A)$, the ring $\hat{A} \otimes_A L$ is regular, for every finite field extension L of $k(P)$.

(b) A is universally catenary.

The completion $\hat{A}$ of an excellent local ring A inherits many good properties from A; basically the completion has all the properties which can be expressed in terms of Serre's conditions, such as the following:

Theorem 1.5. *Let* $(A, \mathbf{m})$ *be an excellent local ring, let* $Q \in \operatorname{Spec}(\hat{A})$, *and* $P = Q \cap A$. *Then the ring* A_P *is regular* (*respectively normal, reduced, Cohen-Macaulay, Gorenstein*) *if and only if the ring* $\hat{A}_Q$ *is regular* (*respectively normal, reduced, Cohen-Macaulay, Gorenstein*).

Corollary 1.6. *For an excellent local ring* $(A, \mathbf{m})$, *the singular locus of* A *is closed in* $\operatorname{Spec}(A)$.

If the ring A is not semilocal, the singular locus is not necessarily closed when the formal fibers of its localizations at prime ideals are geometrically regular. Thus the definition of excellence requires an additional condition.

Definition 1.7. A Noetherian ring A is called *excellent* if

(a) for every maximal ideal $\mathbf{m} \in \operatorname{Spec}(A)$ the local ring $A_{\mathbf{m}}$ is excellent.

(b) The regular locus of A is universally open, that is, for every A-algebra B of finite type, the regular locus of B is open in $\operatorname{Spec}(B)$.

With this definition the class of excellent Noetherian rings is closed under extensions of finite type:

Theorem 1.8. *Let A be an excellent ring and B an A-algebra of finite type. Then B is excellent.*

The proof is nontrivial (see [**8**, (7.8.3)(ii)]). The difficult part is to prove the statement, "Let $\hat{A}$ be a complete local ring and P a prime ideal of $\hat{A}$. Then $\hat{A}_P$ is excellent."

Examples 1.9. (a) Fields, the integers $\mathbf{Z}$, and complete local Noetherian rings are excellent.

(b) The easiest example of a nonexcellent ring can be obtained as follows:

Let K be a field of characteristic $p > 0$, X a variable. Consider the canonical extension:

$$K[X]_{(X)} \longrightarrow K[[X]].$$

Let $\omega \in K[[X]]$ be a power series which is transcendental over $K[X]$. Put

$$C = K(X, \omega^p) \cap K[[X]].$$

The ring C is a discrete valuation ring with completion $\hat{C} = K[[X]]$. However, C is not excellent since the extension of the quotient fields $Q(C) \to K((X))$ is not separable. For an excellent local domain A whose completion $\hat{A}$ is also a domain, the field extension $Q(A) \hookrightarrow Q(\hat{A})$ is always separable. (This last result holds with the much weaker hypothesis that the ring A be Nagata.)

Remarks 1.10. (a) If A is a discrete valuation ring of characteristic 0, then A is excellent.

(b) The discrete valuation ring C of (1.9) is not of finite type over the field K.

(c) Nagata provided the first example of a local Noetherian (nonexcellent) normal domain whose completion is not reduced [**15**, Example 7]. It also provides an example of a two-dimensional regular local ring containing the rationals which fails to be excellent (or even Nagata).

Here is another property of interest for applications of excellent rings:

Theorem 1.11. *Let A be an excellent ring, $I \subseteq A$ an ideal, and $A^* = \widehat{(A, I)}$ the I-adic completion of A. Then A^* is excellent if one of the following conditions is satisfied*:

(a) *the ring A is semilocal, or*

(b) *the rationals are contained in A.*

A proof of this theorem can be found in [**5, 21** and **22**]. In the case where A is not semilocal, the proof uses resolution of singularities. If resolution of singularities holds true in general (currently a matter of some uncertainty), then the theorem is always true, that is, A^* is always excellent.

In the special case where $A^* = K[X_1, \ldots, X_n][[Y_1, \ldots, Y_m]]$, a power series ring over a polynomial ring over a field K, A^* is excellent, even when K has positive characteristic (see [**27**]). In this case, enough derivations on A^* exist to yield Jacobian criteria on the ring, and therefore we get a special proof of the excellence of A^*.

Theorem 1.11 is of interest for questions of the following type: Suppose that a certain class of excellent local rings has property $\mathcal{P}$. Does the completion $\hat{A}$ of A also have property $\mathcal{P}$? Using Theorem 1.11 we can pass from A to $\hat{A}$ in steps of taking completions with respect to principal ideals. Thus, often it is sufficient to show the following statement:

($*$) Let A be an excellent local ring which satisfies property $\mathcal{P}$, and let $s \in \mathbf{m}$ be an element in the maximal ideal of A. Then the (s)-adic completion $\widehat{(A, (s))}$ has property $\mathcal{P}$.

A proof of statement $(*)$ is often technically easier than the statement for $\hat{A}$. One reason is that every ideal in $(\widehat{A,(s)})$ which contains the element s is extended from A. This idea can be used to show that some properties of complete local rings descend to excellent local rings or to certain classes of excellent local rings like excellent local Henselian rings.

2. Henselian rings. In the following let $(A, \mathbf{m}, k)$ be a local Noetherian ring.

Definition 2.1. The ring A is *Henselian* if the following condition, known as Hensel's lemma, is satisfied:

For every monic polynomial $P \in A[X]$ such that, over the residue class field k, the image $\overline{P}$ of P can be expressed as $\overline{P} = \overline{FG} \in k[X]$, with $\overline{F}$ and $\overline{G}$ relatively prime monic polynomials, there exist monic polynomials $F, G \in A[X]$ with $P = FG$ and $F \equiv \overline{F}$, $G \equiv \overline{G} \bmod (\mathbf{m}A[X])$.

Remark 2.2. Henselian rings were first observed in algebraic number theory. Hensel's lemma holds for the ring of p-adic numbers $\widehat{\mathbf{Z}_{(p)}} = \varprojlim \mathbf{Z}/(p^n)$. More generally,

Theorem 2.3. *Let $A = \hat{A}$ be a complete local Noetherian ring. Then A is Henselian.*

Many popular local Noetherian rings fail to be Henselian:

Example 2.4. Let k be a field, $A = k[X]_{(X)}$ the polynomial ring over k. The monic polynomial $P = T^2 + T + X \in A[T]$ is irreducible, but its image

$$\overline{P} = T^2 + T = T(T+1) \in k[T]$$

is reducible and splits into relatively prime factors. Thus A fails to be Henselian.

By Theorem 2.3, every local Noetherian ring can be embedded in a local Henselian ring.

Question 2.5. Is there a "smallest" Henselian ring (in $\hat{A}$) containing a given local Noetherian ring A, and, if so, is this ring unique up to isomorphism?

The answer is "yes," and this "smallest" Henselian ring containing A is called the Henselization of A. We now outline the construction of the Henselization. This requires the notion of an étale morphism:

Definition 2.6. Let $\phi : (A, \mathbf{m}) \to (B, \mathbf{N})$ be a local morphism with B essentially finite over A. Then B is called *étale* over A if the following condition is satisfied:

Suppose that C is an A-algebra, $N \subseteq C$ is an ideal with $N^2 = 0$, and that the following diagram commutes:

$$\begin{array}{ccc} B & \xrightarrow{\bar{u}} & C/N \\ \uparrow{\scriptstyle\phi} & & \uparrow{\scriptstyle\nu} \\ A & \longrightarrow & C \end{array}$$

Then there is a unique A-algebra morphism $u : B \to C$ which lifts $\bar{u}$, that is, $\nu u = \bar{u}$.

B is called an *étale neighborhood* of A if B is étale over A and $A/\mathbf{m} \cong B/\mathbf{N}$ (that is, there is no residue field extension).

Note that Henselian local rings are closed under extensions by étale neighborhoods:

Theorem 2.7. *Let A be a local Henselian ring. Then A is closed under étale neighborhoods, that is, for every étale neighborhood $\phi : A \to B$, we have that $A \cong B$, considered as A-algebras.*

Next we observe that étale extensions are very common:

2.8. *Structure theorem for etale extensions.* Let $\phi : (A, \mathbf{m}) \to (B, \mathbf{N})$ be a local morphism with B essentially finite over A. Then B is étale over A if and only if

$$B \cong (A[X]/(P))_{\mathbf{N}}$$

where

(1) $P \in A[X]$ is a monic polynomial.

(2) $\mathbf{N} \subseteq A[X]$ is a prime ideal with $\mathbf{N} \cap A = \mathbf{m}$.

(3) $P' \notin \mathbf{N}$.

The proof of the structure theorem is quite hard. It is an application of Zariski's main theorem [**20**, Chapter 4].

Using the structure theorem we may define a "representative set" of the étale neighborhoods of A:

$$\Lambda = \left\{ (P, N) \left| \begin{array}{l} P \in A[X] \text{ monic} \\ \mathbf{N} \in \operatorname{Spec}(A[X]), P \in \mathbf{N} \\ P' \notin \mathbf{N}, \mathbf{N} \cap A = \mathbf{m} \\ (A[X]/\mathbf{N})_{\mathbf{N}} = A/\mathbf{m} \end{array} \right. \right\}$$

$$\subseteq A[X] \times \operatorname{Spec}(A[X]).$$

Our goal is to define the Henselization as a direct limit over the set Λ. Let $\lambda_1 = (P_1, \mathbf{N}_1)$ and $\lambda_2 = (P_2, \mathbf{N}_2)$ be elements of Λ, and let $B_1 = (A[X]/(P_1))_{\mathbf{N}_1}$, respectively, $B_2 = (A[X]/(P_2))_{\mathbf{N}_2}$, denote the corresponding étale neighborhoods. Then we define a partial order on Λ by $\lambda_1 \leq \lambda_2$ if and only if there is a local A-algebra morphism $\tau : B_1 \to B_2$. In order to define a direct limit over the set Λ, two conditions must be satisfied. First, the set of A-algebra morphisms between B_1 and B_2 has to be rather small in order to restrict each choice of A-algebra morphisms to one for which "it all fits together." Second, the partially ordered set Λ must be directed, that is, for every pair $\lambda_1, \lambda_2 \in \Lambda$, there must be a third element $\lambda_3 \in \Lambda$ with $\lambda_1 \leq \lambda_3$ and $\lambda_2 \leq \lambda_3$. The following result makes this work:

Theorem 2.9. *Let $B_1 = (A[X]/(P_1))_{\mathbf{N}_1}$ and $B_2 = (A[X]/(P_2))_{\mathbf{N}_2}$, be étale neighborhoods of A. Then*:

(1) *there is at most one A-algebra morphism $\tau : B_1 \to B_2$.*

(2) *There is an element $\lambda_3 = (P_3, \mathbf{N}_3) \in \Lambda$ which contains B_1 and B_2.*

By statement (2), there exists an étale neighborhood $B_3{=}(A[X]/(P_3))_{\mathbf{N}_3}$ and A-algebra morphisms such that the following diagram commutes:

$$\begin{CD} A @>>> (A[X]/(P_1))_{\mathbf{N}_1} \\ @VVV @VVV \\ (A[X]/(P_2))_{\mathbf{N}_2} @>>> (A[X]/(P_3))_{\mathbf{N}_3} \end{CD}$$

Theorem 2.9 shows that the set

$$\{(A[X]/(P))_{\mathbf{N}} \mid (P,\mathbf{N}) \in \Lambda\}$$

is directed in a natural way. The direct limit of this system is defined to be the Henselization of the ring A:

$$A^h = \varinjlim_{\lambda=(P,\mathbf{N})\in\Lambda} (A[X]/(P))_{\mathbf{N}}.$$

We list a few properties of the Henselization:

(a) A^h is the smallest Henselian ring containing A. The Henselization is unique up to isomorphism with respect to this property. That is, if C is a Henselian ring and $\phi : A \to C$ is a local morphism of rings, then ϕ factors through the Henselization of A:

$$\begin{array}{ccc} A & \xrightarrow{\ \phi\ } & C \\ & {\scriptstyle\nu}\searrow & \uparrow{\scriptstyle\tau} \\ & & A^h \end{array}$$

(b) The Henselization is defined similarly for non-Noetherian quasilocal rings. However, a ring A is Noetherian if and only if its Henselization A^h is.

(c) If $(A, \mathbf{m})$ is a local Noetherian ring, then its Henselization A^h is again a local Noetherian ring with maximal ideal $\mathbf{m}A^h$. The Henselization is situated between A and its completion $\hat{A}$ and has the same completion:

$$A \hookrightarrow A^h \hookrightarrow \hat{A} = \widehat{A^h}.$$

(d) The formal fibers of a ring A are geometrically regular if and only if the formal fibers of the Henselization A^h are geometrically regular (see [**9**, Chapter 4]). If A is excellent, then so is its Henselization A^h [**9**, Chapter 4].

Remarks 2.10. (a) The Henselization A^h of a ring A is in general much smaller than the completion $\hat{A}$. In particular, A^h is an algebraic extension of A, whereas its completion $\hat{A}$ is usually of infinite (uncountable) transcendence degree over A.

(b) If A is an excellent normal domain, its Henselization A^h is the "algebraic closure" of A in $\hat{A}$, that is, every element $\omega \in \hat{A} - A^h$ is transcendental over A. The proof uses the fact that any finite normal extension B of A^h is local, since A^h is Henselian, and that the completion $\hat{B}$ is a local normal domain, since A is excellent. The statement also follows trivially if we assume that A^h has the approximation property, a property of local Noetherian rings which we discuss next. Both of these concepts, excellent rings and Henselian rings, are central to this discussion.

3. Artin approximation.

Definition 3.1. Let $(A, \mathbf{m})$ be a local Noetherian ring with completion $(\hat{A}, \hat{\mathbf{m}})$. We say that A has the *approximation property* if, for every ideal $f = (f_1, \dots, f_r) \subseteq A[X_1, \dots, X_n]$ for which the system of equations $f = 0$ has a solution $\hat{a} = (\hat{a}_1, \dots, \hat{a}_n)$ in $\hat{A}^n$, there is an element $a = (a_1, \dots, a_n) \in A^n$ with $f(a) = 0$.

If A has the approximation property, then every system of equations over A which has a solution in the completion $\hat{A}$ is already solvable in A. Moreover, if A has the approximation property and $\hat{a} \in \hat{A}^n$ is a solution for a system of equations $f = 0$ over A, then $\hat{a}$ can be approximated by solutions in A, that is, for every integer $k \in \mathbf{N}$, there is an element $a_{(k)} \in A^n$ such that $f(a_{(k)}) = 0$ and $a_{(k)} \equiv \hat{a} \pmod{\hat{\mathbf{m}}^k \hat{A}^n}$.

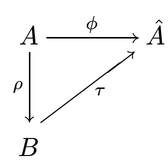
Remark 3.2. Since complete local rings are Henselian, it is easy to see that every ring with the approximation property is Henselian.

M. Artin showed in his celebrated paper [**2**] of 1969 that the following class of rings has the approximation property: Let D be a field or an excellent discrete valuation ring. Then the ring

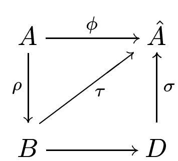
$$A = (D[X_1, \ldots, X_n]_{(\mathbf{m}_D, X_1, \ldots, X_n)})^h$$

has the approximation property. He conjectured in the same paper that excellent Henselian rings have the approximation property. In the following we will refer to this claim as Artin's first conjecture. In the early '80's Artin established an even stronger proposition, referred to here as his second conjecture, which would imply his first conjecture. Before stating this we must introduce another concept:

Let $f = (f_1, \ldots, f_r) \subseteq A[X_1, \ldots, X_n]$ be an ideal. It is easy to see that the system of equations $f = 0$ has a solution in $\hat{A}$ if and only if the canonical morphism $\phi : A \to \hat{A}$ factors through the A-algebra $B = A[X_1, \ldots, X_n]/(f)$:



Suppose that the morphism τ factors through an algebra D which is of finite type over A and smooth over A:



Then, assuming that A is Henselian, there is an A-algebra morphism $\gamma : D \to A$. This last statement follows easily from a well-known property of Henselian rings (see [**3**] for details). Since B maps into D, the existence of a solution of $f = 0$ in A is implied, and we are done.

This yields the following question: Suppose that A is an excellent Henselian local ring, and suppose that there is given an A-algebra B such that the first diagram commutes. Then does the morphism τ always factor through a smooth A-algebra D of finite type? This can be considered as a property of the morphism ϕ and the question can be asked in much more generality. First we need another definition:

Definition 3.3. Let $\phi : A \to B$ be a morphism of Noetherian rings. Then ϕ is called a *regular* morphism if ϕ is flat and if, for all $P \in \operatorname{Spec}(B)$, the fiber ring $B \otimes_A k(P)$ is geometrically regular over $k(P)$, that is, for all finite field extensions $k(P) \subseteq L$, the ring $B \otimes_A L$ is regular.

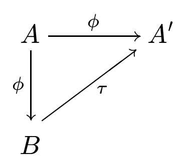
Remark 3.4. Let $(A, \mathbf{m})$ be a Henselian local ring. Then it is well known that A is excellent if and only if the canonical morphism $A \hookrightarrow \hat{A}$ is regular.

Now we are ready to state Artin's second conjecture:

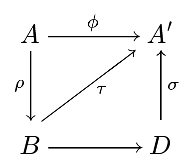
Artin's second conjecture. *Let $\phi : A \to A'$ be a regular morphism of Noetherian rings. Is A' a direct limit of smooth A-algebras of finite type?*

The meaning of this conjecture is exactly what was described above:

Suppose that the morphism ϕ factors through an A-algebra B of finite type:



In order for A' to be a direct limit of smooth A-algebras of finite type every such morphism τ has to factor through a smooth A-algebra D of finite type:



Artin's second conjecture is very powerful. It implies not only his first conjecture but also some parts of the Bass-Quillen conjecture, which has been open for a long time.

Some remarks on the status of Artin's conjectures:

Artin published his famous paper in 1969, in which he established the approximation property for the Henselization of localizations of rings of finite type over a field, respectively over an excellent discrete valuation ring. He had already shown in an earlier (1968) paper that convergent power series rings over the complex numbers possess the approximation property. The proofs in both papers rely heavily on the Weierstrass preparation theorem. In the paper of 1969, Artin also conjectured that excellent Henselian rings have the approximation property. This conjecture seemed inaccessible for a long time, mainly because Artin's proof did not generalize to arbitrary excellent Henselian rings. The Weierstrass preparation theorem is not available over arbitrary excellent Henselian rings. For example, the Weierstrass preparation theorem fails for rings of the type $A = K[X_1, \dots , X_n]_{(X)}[[Y_1, \dots , Y_m]]$ or their Henselizations (K is a field). It seems impossible to extend Artin's proof of the approximation property to this class of Noetherian rings. See [**5**] for a discussion of the case where $n + m = 5$.

In 1985 and 1986, D. Popescu published two papers investigating Artin's second conjecture in the case where the rings contain a field [**17, 18**]; there was considerable discussion about these papers. Later, T. Ogoma published a paper in which he explains the critical parts in Popescu's proof [**16**]. Popescu's proof (and necessarily probably any proof of Artin's second conjecture) distinguishes between the characteristics of the rings involved. The cases where A contains a field of positive characteristic, and where A does not contain a field, require extra work and are considered more difficult than the characteristic zero case.

In 1987 this author published a paper proving Artin's first conjecture for excellent Henselian rings which contain the rationals [**23**].

In 1992 M. Spivakovsky and M. André both made preprints available in which they prove Artin's second conjecture in some important cases. M. André also investigates the case where the rings contain a field; he uses his cohomology for the proof and, as he mentions, some of Popescu's ideas. The proof offered by Spivakovsky is quite different from Popescu's proof. Spivakovsky considers the case where the base ring A contains an arbitrary field as well as some mixed characteristic cases. Although Spivakovsky's proof seems difficult to comprehend to this author, it is generally agreed that the case where the rationals are contained in the base ring is settled.

Finally we note that the converse of Artin's first conjecture is true in all cases. Rings with the approximation property are excellent and Henselian (see [**24**]).

4. An application: The Bass-Quillen conjecture. The approximation theorem is one outstanding application of Artin's second conjecture. Another very important application, the Bass-Quillen conjecture, is quite different in nature.

In 1979 Quillen and Suslin solved Serre's conjecture for projective modules:

Serre's conjecture 4.1. *Let K be a field, and let M be a finitely generated projective $K[X_1, \dots, X_n]$-module. Then M is a free module.*

Actually, Quillen proved an even stronger result, that Serre's conjecture also holds for K a principal ideal domain. Shortly afterwards, the Bass-Quillen conjecture, a generalized form of Serre's conjecture, was introduced:

The Bass-Quillen conjecture 4.2. *Let R be a regular Noetherian ring, and let M be a finitely generated projective module over the polynomial ring $R[X]$. Then M is extended from R, that is, there is a finitely generated projective R-module M_0 such that $M \cong M_0 \otimes_R R[X]$.*

In many cases, for example, if the ring R contains a field, the Bass-Quillen conjecture becomes a corollary of Artin's second conjecture. This can be seen as follows:

Suppose that R contains a field, and let $P \subseteq R$ be the prime field. Since R is a regular ring and P is a perfect field, the embedding $P \hookrightarrow R$ is a regular morphism. Thus, by Artin's second conjecture, R is a direct limit of smooth P-algebras D of finite type. By investigating a free resolution of the $R[X]$-module M, it can be seen that there is a finite type P-subalgebra B of R and a projective $B[X]$-module M_0 with $M \cong M_0 \otimes_{B[X]} R[X]$. The embedding $B \hookrightarrow R$ factors through a finite type smooth P-algebra D, and we obtain a finitely generated projective $D[X]$-module M_1 with $M \cong M_1 \otimes_{D[X]} R[X]$. The smoothness of D over

P is equivalent to D being a regular ring. Thus it is enough to show the conjecture in the case where the ring R is a regular finitely generated algebra over a field k. The validity of the Bass-Quillen conjecture in this case has been shown by Lindel [**13**].

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